1 Chapter 2

1.1 Tensor product

1. Exercise 1. Show that $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is zero if m, n are coprime.

   Exercise 1 solution. It follows quickly from the definition of the tensor product that $0 \otimes y = x \otimes 0 = 0$ in any tensor product of modules.

   Another useful lemma is that if M is a cyclic A-module, i.e. if there is a $g \in M$ such that all $x \in M$ have the form ag for some $a \in A$ (in which case g is said to generate M), and N is a cyclic A-module generated by h, then $M \otimes_A N$ is a cyclic A-module and it is generated by $g \otimes h$. This is because for any $x \otimes y \in M \otimes N$, we have $x = ag, y = bh$ for $a, b \in A$, and then $x \otimes y = ag \otimes bh = (ab)(g \otimes h)$, so that any element of $M \otimes_A N$ of the form $x \otimes y$ can be written $r(g \otimes h)$ for some $r \in A$. But then for any $\sum_i x_i \otimes y_i \in M \otimes_A N$, we can write $\sum_i x_i \otimes y_i = \sum_i r_i(g \otimes h) = (\sum_i r_i)(g \otimes h)$.

   Because of this lemma, since $\mathbb{Z}/m\mathbb{Z}$ and $\mathbb{Z}/n\mathbb{Z}$ are cyclic, generated by 1, $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is cyclic and generated by $1 \otimes 1$. But since m, n are coprime, there exist integers $a, b \in \mathbb{Z}$ such that $am + bn = 1$, and then

   $$1 \otimes 1 = (am + bn)(1 \otimes 1) = am(1 \otimes 1) + bn(1 \otimes 1) = a(m \otimes 1) + b(1 \otimes n) = a0 + b0 = 0$$

   Thus $(\mathbb{Z}/m\mathbb{Z}) \otimes_{\mathbb{Z}} (\mathbb{Z}/n\mathbb{Z})$ is generated by 0, so it is 0.

2. Exercise 2. Let A be a ring, $\mathfrak{a}$ an ideal, and M an A-module. Show that $A/\mathfrak{a} \otimes_A M$ is isomorphic as an A-module to $M/\mathfrak{a}M$.

   Exercise 2 solution. We have an exact sequence of A-modules:

   $$0 \to \mathfrak{a} \to A \to A/\mathfrak{a} \to 0$$

   By right-exactness of the tensor product, tensoring with M yields the exact sequence

   $$\mathfrak{a} \otimes_A M \to A \otimes_A M \to (A/\mathfrak{a}) \otimes_A M \to 0$$

   Now $A \otimes_A M$ is canonically isomorphic to M via the map $a \otimes m \mapsto am$ (linearly extended to all of $A \otimes M$), with inverse given by $m \mapsto 1 \otimes m$. So (via this identification) we can write

   $$\mathfrak{a} \otimes_A M \to M \to (A/\mathfrak{a}) \otimes_A M \to 0$$

   What is the image of $\mathfrak{a} \otimes_A M$ inside M? It is finite sums of elements of the form am with $a \in \mathfrak{a}$. This is precisely the submodule $\mathfrak{a}M$. Since the sequence is exact, this is also the kernel of the map $M \to A/\mathfrak{a} \otimes_A M$, which is surjective. Thus $(A/\mathfrak{a}) \otimes_A M$ is canonically isomorphic to $M/\mathfrak{a}M$.

3. Exercise 3. Prove that if A is a local ring, M and N are finitely generated A-modules, and $M \otimes_A N = 0$, then one of M or N is zero.

   Exercise 3 solution. By problem 2, we have $M \otimes_A A/\mathfrak{m} = M/\mathfrak{m}M$, so M surjects onto $M \otimes_A k$ (where $k = A/\mathfrak{m}$). Similarly for N and $N \otimes_A k$. Thus

   $$M \otimes_A N \to (M \otimes_A k) \otimes_A (N \otimes_A k)$$

   is surjective. Now the module on the right is really a k-module and the tensor product is really over k, since the tensor product of k-modules over A is really over k since factors that pull across are in equivalence classes mod $\mathfrak{m}$. Thus if $M \otimes_A N$ is zero, then

   $$(M \otimes_A k) \otimes_k (N \otimes_A k)$$

   is zero. This is a tensor product of k-vector spaces of finite dimensions (since M, N are finitely generated), say m and n. Then this has dimension $mn = 0$. So $m = 0$ or $n = 0$ since $\mathbb{Z}$ is a domain! Then $M \otimes_A k = M/\mathfrak{m}M = 0$ (or similarly for N) implying that $M = \mathfrak{m}M$ (or likewise for N). Now the Nakayama lemma applies since M (or N) is finitely generated and $\mathfrak{m}$ is the Jacobson radical of A, to show that $M = 0$ (or $N = 0$). This completes the argument.

1.2 Modules

1. Exercise 7. Let $\mathfrak{p} \lhd A$ be a prime ideal. Show that $\mathfrak{p}[x]$ is a prime ideal of $A[x]$. If $\mathfrak{m}$ is maximal in A, is $\mathfrak{m}[x]$ maximal in $A[x]$?

   Exercise 7 solution. Define the ring homomorphism $A[x] \to (A/\mathfrak{p})[x]$ in the obvious way. The kernel of this homomorphism is $\mathfrak{p}[x]$, because clearly any polynomial with all coefficients in $\mathfrak{p}$ gets knocked out, and any polynomial with any coefficient not in $\mathfrak{p}$ does not get knocked out. Because $A/\mathfrak{p}$ is a domain, $(A/\mathfrak{p})[x]$ is a domain, and it follows that the kernel $\mathfrak{p}[x]$ of this homomorphism is prime.

   It does not follow from $\mathfrak{m}$ maximal that $\mathfrak{m}[x]$ will be maximal. For a counterexample, take $A = \mathbb{Z}$, $\mathfrak{m} = (2) \lhd \mathbb{Z}$. Then $\mathfrak{m}[x] = (2) \lhd \mathbb{Z}[x]$, but this is properly contained in $(2, x)$. In terms of our argument above, in general $\mathfrak{m}[x]$ won't be maximal in $A[x]$ because though $A/\mathfrak{m}$ is a field, $(A/\mathfrak{m})[x]$ is not.

2. Exercise 9. Let $0 \to M' \to M \to M'' \to 0$ be an exact sequence of A-modules. If M', M'' are finitely generated, so is M.

   Exercise 9 solution. The generators of M' and any representatives of the generators for M'' form a set of generators for M.

3. Exercise 10. Let A be a ring, $\mathfrak{a}$ an ideal contained in the Jacobson radical of A; let M be an A-module and N a finitely generated A-module; and let $u : M \to N$ be a homomorphism. Show that if the induced homomorphism $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then u is surjective.

   Exercise 10 solution. If $M/\mathfrak{a}M \to N/\mathfrak{a}N$ is surjective, then every coset of $\mathfrak{a}N$ contains an element of $\operatorname{im} u$, thus $N = \operatorname{im} u + \mathfrak{a}N$. Now apply Nakayama's lemma, in the form of Corollary 2.7, to N, to conclude that $N = \operatorname{im} u$.

4. Exercise 11. Let A be a nonzero ring. Show that $A^m \cong A^n \Rightarrow m = n$. Show that if $\phi : A^m \to A^n$ is surjective, then $m \geq n$. If $\phi : A^m \to A^n$ is injective, is m necessarily $\leq n$?

   Exercise 11 solution. It is much easier to prove that isomorphism implies $m = n$ and surjectivity implies $m \geq n$ than that injectivity implies $m \leq n$, but this last is also true. Here are proofs:

   An isomorphism $A^m \to A^n$ fits into an exact sequence

   $$0 \to A^m \to A^n \to 0$$

   Now pick any maximal ideal $\mathfrak{m}$ of A; then $A/\mathfrak{m}$ is a field. Right-exactness of the tensor product, together with the fact that $A^m \otimes_A A/\mathfrak{m} = (A/\mathfrak{m})^m$, which is a consequence of problem 2 but is not hard to see directly, gives us an exact sequence

   $$0 \to (A/\mathfrak{m})^m \to (A/\mathfrak{m})^n \to 0$$

   which is an isomorphism of finite-dimensional vector spaces over the field $A/\mathfrak{m}$. Thus the dimensions must be equal: $m = n$.

   If $\phi : A^m \to A^n$ is surjective, we can repeat this same argument except on the exact sequence

   $$\ker \phi \to A^m \to A^n \to 0$$

   obtaining a surjection of $(A/\mathfrak{m})^m$ onto $(A/\mathfrak{m})^n$, and we can conclude $m \geq n$.

   If $\phi : A^m \to A^n$ is injective, we do not have access to the same argument, because the tensor product is not left-exact, precisely because it does not preserve injectivity. We need another argument.

   Suppose for a contradiction that $\phi : A^m \to A^n$ is injective but has $m > n$. Compose with the embedding of A^n in the first n coordinates of A^m; thus ϕ becomes an endomorphism of A^m, and because $m > n$, its image is always zero in the last coordinate.

By Proposition 2.4 (taking $\mathfrak{a}$ to be all of A), ϕ satisfies a monic polynomial in the ring of endomorphisms of A^m:

$$\phi^k + a_1\phi^{k-1} + \cdots + a_k = 0$$

Actually it obeys an ideal of such polynomials; let this be one of minimal degree. Then, since ϕ is injective, we cannot have $a_k = 0$. If we did, it would be divisible by ϕ, and we would have

$$\phi \circ (\phi^{k-1} + a_1\phi^{k-2} + \dots) = 0$$

but $\phi^{k-1} + a_1\phi^{k-2} + \dots$ is not identically zero as an operator, by the minimality assumption we just made. Thus there exists $x \in A^m$ with $(\phi^{k-1} + a_1\phi^{k-2} + \dots)x \neq 0$. But then applying ϕ to $(\phi^{k-1} + a_1\phi^{k-2} + \dots)x$ yields zero. But this contradicts the assumption that ϕ is injective. So $a_k \neq 0$.

But now we have a contradiction. Let $\psi = \phi^k + a_1\phi^{k-1} + \cdots + a_k$. ψ is supposed to be identically zero as an endomorphism. However, because $\operatorname{im}\phi$ is all zeros in the last coordinate, if ψ is applied to any element of A^m that has 1 in the last coordinate, all the ϕ terms will contribute nothing to this coordinate and it will come out equal to a_k, which we just showed is nonzero. Thus ψ cannot be identically zero as an endomorphism. This contradiction proves that $m \leq n$.

5. Exercise 12. Let M be a finitely generated A-module and suppose that $\phi : M \to A^n$ is surjective. Show that $\ker\phi$ is finitely generated.

   Exercise 12 solution. Let $u_1, \dots, u_n$ be preimages for $e_1, \dots, e_n$. (These exist by surjectivity.) Let U be the submodule of M generated by $u_1, \dots, u_n$. We have

$$\phi\left(\sum a_i u_i\right) = \sum a_i e_i$$

   for $a_i \in A$, by definition of an A-module homomorphism. Since A^n is free, every choice of distinct a_i's yields a different image on the right. Thus every distinct choice of a_i's yields a distinct element of U, and $\phi|_U$ is an isomorphism. We can identify A^n with U and think of ϕ as an endomorphism, and then we have $\phi^2 = \phi$.

   Whenever this condition is met on a map ϕ, its image U is a direct summand of M. Indeed, define a new map ψ by $\psi(x) = x - \phi(x)$. Linearity of ϕ implies that of ψ. Meanwhile we have $\psi^2(x) = \psi(x-\phi(x)) = x-\phi(x)-(\phi(x)-\phi^2(x)) = x-\phi(x)-0 = \psi(x)$, so $\psi^2 = \psi$ too, and for $x \in U$ we have $\psi(x) = x - \phi(x) = \phi(y) - \phi^2(y) = 0$ for some y, so that $\ker\psi \supset U$. Conversely, if $y \in \ker\psi$, then by definition $y - \phi(y) = 0$, so that $y \in U$, so $\ker\psi = U$. Since this follows from the relations $\phi^2 = \phi, \psi^2 = \psi$ and $\phi(x) + \psi(x) = x, \forall x \in M$, which are symmetric in ϕ, ψ, the same arguments will give us, if $V = \operatorname{im}\psi$, $\ker\phi = V$. We have $x \mapsto (\phi(x), \psi(x))$ is a homomorphism from M to $U \oplus V$, and it is inverted by $(u, v) \mapsto u + v$, so it is an isomorphism. We have proven:

An idempotent endomorphism is a projection to a direct summand.

Anyway, let $m_1, \dots, m_k$ be generators for M. Then since ψ is a surjective homomorphism onto $V = \ker\phi$, $\psi(m_1), \dots, \psi(m_k)$ generate V.

1.3 Direct Limit

1. Exercise 14. A poset I is a *directed set* if every pair of elements has a common upper bound. Let A be a ring, I a directed system, and $(M_i)_{i\in I}$ a family of A-modules indexed by I. Suppose that for every pair i, j with $i \le j$, there is a homomorphism $\mu_{ij} : M_i \to M_j$, such that μ_{ii} is the identity for all $i \in I$, and the homomorphisms are compatible with the structure of I: if $i \le j \le k$, then $\mu_{ik} = \mu_{jk} \circ \mu_{ij}$. Then $\mathbf{M} = (M_i, \mu_{ij})$ is a *direct system* over I. It has a *direct limit*

   $$M = \varinjlim M_i$$

   which is an A-module, defined as the quotient of $\bigoplus_{i\in I} M_i$ by the submodule generated by identifying every element of every M_i with its image under any of the μ_{ij}. (Call this submodule D.) Each M_i has a canonical homomorphism μ_i to M by restricting the canonical homomorphism on the whole direct sum to M_i. Call the whole canonical homomorphism μ.

   Exercise 14 solution. This exercise does not really give the reader anything to do .

   Here are some examples of this construction (requiring varying amounts of background):

   (a) Let $A = \mathbb{Z}$, let $I = \mathbb{N}$ with the canonical ordering, and let each M_i be isomorphic to $\mathbb{Z}$, with $\mu_{i(i+1)}$ being multiplication by 2 every time. Then $\varinjlim M_i$ is, as an abelian group i.e. a $\mathbb{Z}$-module, the rational numbers with denominators a power of 2. Each M_i can be thought of as $\mathbb{Z}/2^{i-1}$.

   (b) Similarly, take the same A and I and again let each $M_i \cong A$, but this time order I by divisibility, so $i \le j$ means $i \mid j$. For $i \mid j$, let μ_{ij} be multiplication by j/i. Now $\varinjlim M_i$ is the additive group of $\mathbb{Q}$, and M_i can be thought of as $\mathbb{Z}/i$.

   (c) Let M be a module; let I be the family of finitely generated submodules of M, ordered by inclusion. Then $i \le j$ means $i \subset j$; let μ_{ij} be the embedding $i \hookrightarrow j$. Then $M_i = i$, and $\varinjlim M_i = M$. See problem 17.

   (d) The stalk of a sheaf of functions over a point is the direct limit of the ring of functions on each open set containing the point, ordered by reverse inclusion of the sets. Here we have a topological space X with a sheaf of functions $\mathcal{O}_X$, and a point $p \in X$. I is the family of open sets of X containing p, ordered by reverse inclusion. $M_i = \mathcal{O}_X(i)$ is the family of functions on the open set i. $j \ge i$ means $j \subset i$, and μ_{ij} is the restriction map restricting functions from i to j; the kernel is the set of functions that are zero on j. $\varinjlim \mathcal{O}_X(i) = \mathcal{O}_X(p)$ is, by definition, the stalk over p.

2. Exercise 15. Show that every element of $\varinjlim M_i$ can be written $\mu_i(x_i)$ for some $x_i \in M_i$ for some $i \in I$. ("Everybody in the direct limit is represented by somebody in one of the M_i's.") Also show that if $\mu_i(x_i) = 0$, then there exists $j \ge i$ such that $\mu_{ij}(x_i) = 0$ in M_j. ("If you are zero in the direct limit, you were zero in one of the M_i's.")

   Exercise 15 solution. Prima facie, each element of $M = \varinjlim M_i$ is a sum of finitely many elements of the form $\mu_i(x_i)$ for varying i's. We will induct on the number of terms in the sum. The key case is two. Say $x \in M$ equals $\mu_i(x_i) + \mu_j(x_j')$. Because I is a directed system, we have $k \ge i, j$. Consider $y_k = \mu_{ik}(x_i)$ and $y_k' = \mu_{jk}(x_j')$. By the construction of M, y_k and x_i have the same images in M and so do y_k' and x_j'. Because $y_k, y_k' \in M_k$, we can add them: $x_k = y_k + y_k'$. I claim that $\mu_k(x_k) = \mu_i(x_i) + \mu_j(x_j')$ in M. Indeed, $\mu_k(x_k) = \mu(x_k) = \mu(y_k + y_k') = \mu(y_k) + \mu(y_k') = \mu(x_i) + \mu(x_j') = \mu_i(x_i) + \mu_j(x_j') = x$. So if x has a representation in terms of two $\mu_i(x_i)$'s, then it has a representation in terms of one. A sum with arbitrary finite number of terms follows immediately by induction.

   Proving that if $\mu_i(x_i) = 0$ in M then there exists $j \ge i$ such that $\mu_{ij}(x_i) = 0$ in M_j takes substantially more work but the work pays off in a richer understanding of the direct sum construction:

   We will define a prima facie different A-module M^* from the union (rather than the direct sum) of the M_i's. We will then show it is isomorphic to M. We will answer the question by reasoning about M^*.

   M^* is defined as follows: start with the set $\bigcup_{i \in I} M_i$, and quotient by the relation that $x_i \in M_i$ and $x_j \in M_j$ are equivalent if there is some k lying over i, j such that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. Represent classes in M^* by $[x_i]$. Now impose a module structure on M^*: define $a[x_i] = [ax_i]$. It is clear that this is well-defined: if $x_i \sim x_j$ then $\exists k$ such that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. In which case, $a\mu_{ik}(x_i) = a\mu_{jk}(x_j)$, so $\mu_{ik}(ax_i) = \mu_{jk}(ax_j)$, so $ax_i \sim ax_j$. Now define $[x_i] + [x_j] = [\mu_{ik}(x_i) + \mu_{jk}(x_j)]$ for some k lying over i, j. (Here we are making heavy use of the fact that I is a directed set.) To see that this doesn't depend on the choice of k, consider that for any two k, k''s lying over i, j, there is m lying over both of them, and that we have

$$\mu_{km}\left(\mu_{ik}(x_i) + \mu_{jk}(x_j)\right) = \mu_{im}(x_i) + \mu_{jm}(x_j) = \mu_{k'm}\left(\mu_{ik'}(x_i) + \mu_{jk'}(x_j)\right)$$

   To see that it doesn't depend on the choice of representatives for $[x_i], [x_j]$ consider that if $x_i \sim x_k$ and $x_j \sim x_l$, i.e. there exist p above i, k and q above j, l such that $\mu_{ip}(x_i) = \mu_{kp}(x_k)$ and $\mu_{jq}(x_j) = \mu_{lq}(x_l)$, and r, s lie above i, j and k, l, respectively, we can choose t lying above all these things and then

$$\begin{aligned}\mu_{rt}\left(\mu_{ir}(x_i)+\mu_{jr}(x_j)\right)&=\mu_{it}(x_i)+\mu_{jt}(x_j)\\&=\mu_{pt}(\mu_{ip}(x_i))+\mu_{qt}(\mu_{jq}(x_j))\\&=\mu_{pt}(\mu_{kp}(x_k))+\mu_{qt}(\mu_{lq}(x_l))\\&=\mu_{kt}(x_k)+\mu_{lt}(x_l)\\&=\mu_{st}\left(\mu_{ks}(x_k)+\mu_{ls}(x_l)\right)\end{aligned}$$

Thus, $[x_i]+[x_j]\sim[x_k]+[x_l]$. We have shown that addition is well-defined on M^*; it now has an A-module structure.

There is a natural map from M^* into the direct limit M defined by Atiyah-MacDonald: $\phi: M^*\to M=\bigoplus_{i\in I}M_i/D$ given by $\phi([x_i])=\mu(x_i)=\mu_i(x_i)$. We claim it is an isomorphism. There are several things to check:

(A) It is well-defined. This is the claim that if $x_i\sim x_j$, then $\mu(x_i)=\mu(x_j)$. Indeed, D contains both $x_i-\mu_{ik}(x_i)$ and $x_j-\mu_{jk}(x_j)$ for any x_i,x_j and k lying over i,j. So if $\mu_{ik}(x_i)=\mu_{jk}(x_j)$, it also contains $x_i-\mu_{ik}(x_i)-[x_j-\mu_{jk}(x_j)]=x_i-x_j$. Thus if x_i,x_j are equivalent in M^*, their images in M are equivalent: $\mu(x_i)-\mu(x_j)=\mu(x_i-x_j)=0$.

(B) It is an A-module homomorphism. Compatibility with scalar multiplication is obvious. Compatibility with addition is as follows:

$$\begin{aligned}\phi([x_i])+\phi([x_j])&=\mu(x_i)+\mu(x_j)\\&=\mu_i(x_i)+\mu_j(x_j)\\&=\mu_k(\mu_{ik}(x_i))+\mu_k(\mu_{jk}(x_j))\\&=\mu\left(\mu_{ik}(x_i)+\mu_{jk}(x_j)\right)\\&=\phi\left([\mu_{ik}(x_i)+\mu_{jk}(x_j)]\right)\\&=\phi([x_i]+[x_j])\end{aligned}$$

where k is some element of I lying over i,j.

(C) It is injective and surjective. Surjectivity was essentially what was proven in the first part of this problem, but we can get both it and injectivity out of the same device here. Consider the diagram

$$\begin{array}{ccc} M^* & \rightarrow & M\\ & \nwarrow & \uparrow\\ & & \bigoplus_{i\in I}M_i\end{array}$$

where ϕ is the map $M^*\to M$, the up arrow is the canonical homomorphism with kernel D, which we will call π, and the diagonal arrow is defined by

$$\sum_i x_i\mapsto\sum_i[x_i]$$

Let us call this map ψ. It is more or less immediate that ψ is surjective and that the diagram commutes, i.e. $\pi = \phi \circ \psi$. Since π is surjective, commutativity of the diagram implies that ϕ is surjective. This proves surjectivity. Commutativity of the diagram also implies that $\ker \psi \subset \ker \pi = D$. However, by a straightforward calculation we also have $\ker \psi \supset D$. For $\ker \psi$ includes all the generators of D: such a generator has the form $x_i - \mu_{ij}(x_i)$, but taking $k = j$ we have $\mu_{ij}(x_i) = \mu_{jj}(\mu_{ij}(x_i))$ since $\mu_{jj} = \text{id.}$. So x_i and $\mu_{ij}(x_i)$ represent the same class in M^*, and we have

$$\psi(x_i - \mu_{ij}(x_i)) = [x_i] - [\mu_{ij}(x_i)] = 0$$

Since $\ker \psi$ contains all D's generators, it contains D. This proves equality. Thus, M^* is isomorphic to $M = \bigoplus_{i \in I} M_i / D$, and ϕ is the isomorphism. (To spell out the injectivity argument, if ϕ had a nontrivial kernel, then since ψ is surjective, it would correspond to a submodule of $\bigoplus M_i$ properly containing $\ker \psi = D$; thus the composed map $\phi \circ \psi$ would have a kernel bigger than D. But we know the kernel is exactly D because this composed map is π.)

Anyway, this isomorphism gets the result of the problem with almost no additional work. We may as well move the conversation to M^*, and here, by definition of M^*, $[x_i] = [0]$ if and only if there exists j such that $\mu_{ik}(x_i) = \mu_{jk}(0) = 0$ in M_k. This is what was to be shown.

3. Exercise 16. Show that the direct limit has, and is characterized up to isomorphism by, the following universal property. Let N be an A-module and for each $i \in I$ let $\alpha_i : M_i \to N$ be an A-module homomorphism such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \leq j$. Then there exists a unique homomorphism $\alpha : M \to N$ such that $\alpha_i = \alpha \circ \mu_i$ for all $i \in I$.

   Exercise 16 solution. Let N, α_i be such a module-and-homomorphisms.

   First we show that for each $i \in I$, α_i factors through μ_i. To prove this, we have to show that $\ker \mu_i \subset \ker \alpha_i$. But this is clear. If $x_i \in M_i$ is in $\ker \mu_i$, this means by the last problem that there exists j so that $\mu_{ij}(x_i) = 0$. But since $\alpha_i = \alpha_j \circ \mu_{ij}$, we have $\alpha_i(x_i) = \alpha_j(\mu_{ij}(x_i)) = \alpha_j(0) = 0$, so that $x_i \in \ker \alpha_i$. Thus, $\alpha_i : M_i \to N$ factors through μ_i, i.e. there exists a unique homomorphism, call it β_i, from $\operatorname{im} \mu_i \subset M$ to N, such that $\alpha_i = \beta_i \circ \mu_i$.

   Secondly, we show that the β_i's all agree where they overlap. In other words, if $x \in M$ is in both $\operatorname{im} \mu_i$ and $\operatorname{im} \mu_j$, then $\beta_i(x) = \beta_j(x)$ in N. We see this as follows: $x \in \operatorname{im} \mu_i \cap \mu_j$ implies there exists $x_i \in M_i$ and $x_j \in M_j$ such that $\mu_i(x_i) = \mu_j(x_j) = x$. But then, by our work in the last problem, this means that there exists k over i, j such that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. Then

$$\alpha_i(x_i) = \alpha_k \circ \mu_{ik}(x_i) = \alpha_k \circ \mu_{jk}(x_j) = \alpha_j(x_j)$$

in N. But the construction of the β_i's means that $\alpha_i(x_i) = \beta_i \circ \mu_i(x_i) = \beta_i(x)$, and similarly for x_j, thus

$$\beta_i(x) = \alpha_i(x_i) = \alpha_j(x_j) = \beta_j(x)$$

So the β_i's agree where they overlap. Since by our work in the last problem, the $\operatorname{im}\mu_i$'s cover M, this means that the β_i's give us a well-defined map from M to N. Call this map α. We claim it is the desired homomorphism $M \to N$.

There are two things to check: first, that it is indeed an A-module homomorphism. Second, that it has the desired property that $\alpha_i = \alpha \circ \mu_i$ for all i.

To see it is an A-module homomorphism, take any two elements $x, y \in M$; since the $\operatorname{im}\mu_i$'s cover M by the last problem, we may assume $x \in \operatorname{im}\mu_i$ and $y \in \operatorname{im}\mu_j$ for some $i, j \in I$. Take a $k \in I$ over i, j; then x, y are both in $\operatorname{im}\mu_k$ since $\mu_k \circ \mu_{ik} = \mu_i$ implies $\operatorname{im}\mu_k \supset \operatorname{im}\mu_i$ and similarly $\mu_k \circ \mu_{jk} = \mu_j$ implies $\operatorname{im}\mu_k \supset \operatorname{im}\mu_j$. But β_k is then a homomorphism from the submodule $\operatorname{im}\mu_k$ of M containing both x and y, to N, and it is the restriction of α to this submodule by construction of α. Thus, for $a, b \in A$, we have $ax + by \in \operatorname{im}\mu_k \subset M$, thus

$$\alpha(ax + by) = \beta_k(ax + by) = a\beta_k(x) + b\beta_k(y) = a\alpha(x) + b\alpha(y)$$

This establishes that α is a homomorphism of A-modules.

To see that $\alpha_i = \alpha \circ \mu_i$, consider only that α's restriction to $\operatorname{im}\mu_i$ is β_i by the construction of α, so $\alpha \circ \mu_i = \beta_i \circ \mu_i = \alpha_i$, by the construction of β_i.

This establishes that the unique homomorphism α exists from the direct limit $\varinjlim M_i = M$ to N.

For uniqueness up to isomorphism, let M' be a second A-module, and μ_i' a second set of homomorphisms $M_i \to M'$, such that for all A-modules N and homomorphisms $\alpha_i : M_i \to N$ such that $\alpha_i = \alpha_j \circ \mu_{ij}$ whenever $i \le j$, there exists a unique homomorphism $\alpha' : M' \to N$ such that $\alpha_i = \alpha' \circ \mu_i'$. In particular, taking $N = M$, we find that there exists a unique homomorphism $\mu' : M' \to M$ such that for each μ_i we have $\mu_i = \mu' \circ \mu_i'$.

I claim μ' is an isomorphism. Indeed, by M's universal property we get μ mapping $M \to M'$, and they are clearly inverses. For the composition $\mu \circ \mu'$ is a map $M' \to M'$ with the specified properties, thus it must be the unique one (using M''s unversal property and taking $N = M'$), but meanwhile the identity also satisfies the properties, so we must have $\mu \circ \mu' = \text{id}.$; and similarly for $\mu' \circ \mu$.

4. Exercise 17. Let $\{M_i\}_{i \in I}$ be a family of submodules of an A-module M such that for any $i, j \in I$ there exists $k \in I$ such that $M_i + M_j \subset M_k$. Define $i \le j$ if $M_i \subset M_j$. In this way I becomes a directed set. Define μ_{ij} as the inclusion map of M_i in M_j. In this way the M_i, μ_{ij} become a direct system. Show that

$$\varinjlim M_i = \sum M_i = \bigcup M_i$$

Deduce that in particular, any A-module is the direct limit of its finitely generated submodules.

Exercise 17 solution. Since for any i, j there exists k such that $M_i + M_j \subset M_k$, $\bigcup M_i$ is closed under sums, so that $\sum M_i \subset \bigcup M_i \subset \sum M_i$ so we have equality between these and it just needs to be shown that they are the same as the direct limit.

But this is clear from our work in problem 15, where it is shown that the direct limit is the union modulo equality in some common upper bound module. Here, this translates to equality in M: given $x_i \in M_i$ and $x_j \in M_j$, if there exist k, μ_{ik}, μ_{jk} such that $\mu_{ik}(x_i) = \mu_{jk}(x_j)$, then since all the μ_{ij} are inclusions, we must have $x_i = x_j$ as elements of $M_k \subset M$. Conversely, if x_i and x_j are equal as elements of M, then since there exists k with $M_i + M_j \subset M_k$ we have $\mu_{ik}(x_i) = \mu_{jk}(x_j)$. So the direct limit is precisely the union here.

Now let M be any A-module. The set of finitely generated submodules M_i clearly satisfies the above, since the sum of finitely generated modules is finitely generated. Since every element of M is contained in some finitely generated submodule (e.g. the cyclic module generated by itself), we have $M = \bigcup M_i$, and the above result applies.

5. Exercise 18. Let $\mathbf{M} = (M_i, \mu_{ij})$ and $\mathbf{N} = (N_i, \nu_{ij})$ be two different direct systems of A-modules over the same directed set I. A *homomorphism* $\mathbf{\Phi} : \mathbf{M} \to \mathbf{N}$ *of direct systems* is defined in the natural way. It is a family of module homomorphisms $\phi_i : M_i \to N_i$ that commute with the μ_{ij}, ν_{ij}. I.e., given i, j with $i \le j$, ϕ_i, ϕ_j satisfy $\nu_{ij} \circ \phi_i = \phi_j \circ \mu_{ij}$. Show that a homomorphism of direct systems $\mathbf{\Phi}$ induces a unique homomorphism $\phi = \varinjlim \phi_i$ of the direct limits that commutes with the projections μ_i, ν_i, i.e. such that for all i, $\phi \circ \mu_i = \nu_i \circ \phi_i$.

Exercise 18 solution. Let $N = \varinjlim N_i$. Let $\alpha_i = \nu_i \circ \phi_i$. Then for each $i \in I$, α_i maps M_i to N such that for any $j \ge i$ we have

$$\begin{aligned} \alpha_i &= \nu_i \circ \phi_i \\ &= \nu_j \circ \nu_{ij} \circ \phi_i \\ &= \nu_j \circ \phi_j \circ \mu_{ij} \\ &= \alpha_j \circ \mu_{ij} \end{aligned}$$

so that by the universal property of $M = \varinjlim M_i$ discussed in the last problem, we get a unique homomorphism $\alpha : M \to N$ satisfying $\alpha_i = \alpha \circ \mu_i$ for each $i \in I$. Let $\phi = \alpha$. Then we have

$$\phi \circ \mu_i = \alpha_i = \nu_i \circ \phi_i$$

so that ϕ so defined has the desired property.